Introduction to Abstract Algebra

with Applications to Social Systems

Women in Mathematics

SWIM

Summer Workshop in Mathematics

FOR HIGH SCHOOL STUDENTS

Princeton

Course II Lecture Notes 6 of 7

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Course Resource

Gallian

Contemporary Abstract Algebra by Joseph A. Gallian

Group

Definition

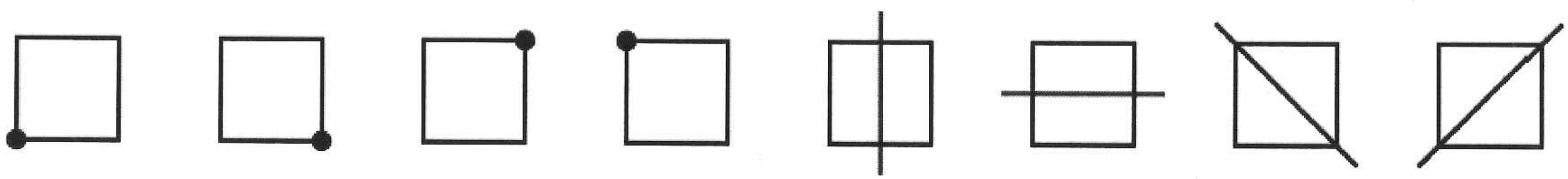
Let *G* be a nonempty set together with a binary operation (usually called multiplication). We say *G* is a *group* under this operation if the following properties are satisfied:

1. Closure: For all *a*, *b* in *G*, *ab* is also in *G*.
2. Identity: There is an element in *G* (called the *identity*) such that $ae = ea = a$ for all *a* in *G*.
3. Inverses: For each element *a* in *G*, there is an element *b* in *G* (called an *inverse* of a) such that $ab = ba = e$.
4. Associativity: The operation is associative; that is, $(ab)c = a(bc)$ for all *a*, *b*, *c* in *G*.

Symmetries of the Square

Description

Suppose we remove a square region from a plane, move it in some way, then put the square back into the space it originally occupied. Describe all possible ways in which this can be done.



R0 R1 R2 R3 M1 M2 D1 D2

Symmetries of the Square

Cayley Table

	R0	R1	R2	R3	M1	M2	D1	D2
R0	R0	R1	R2	R3	M1	M2	D1	D2
R1	R1	R2	R3	R0	D1	D2	M2	M1
R2	R2	R3	R0	R1	M2	M1	D2	D1
R3	R3	R0	R1	R2	D2	D1	M1	M2
M1	M1	D2	M2	D1	R0	R2	R3	R1
M2	M2	D1	M1	D2	R2	R0	R1	R3
D1	D1	M1	D2	M2	R1	R3	R0	R2
D2	D2	M2	D1	M1	R3	R1	R2	R0

Properties of Groups

Subgroup

Definition: If a subset H of a group G is itself a group under the operation of G, we say that H is a subgroup of G.

Symmetries of the Square

Cayley Table

	R0	R1	R2	R3	M1	M2	D1	D2
R0	R0	R1	R2	R3	M1	M2	D1	D2
R1	R1	R2	R3	R0	D1	D2	M2	M1
R2	R2	R3	R0	R1	M2	M1	D2	D1
R3	R3	R0	R1	R2	D2	D1	M1	M2
M1	M1	D2	M2	D1	R0	R2	R3	R1
M2	M2	D1	M1	D2	R2	R0	R1	R3
D1	D1	M1	D2	M2	R1	R3	R0	R2
D2	D2	M2	D1	M1	R3	R1	R2	R0

Subgroup Tests

One-Step Subgroup Test

Theorem: Let G be a group and H a nonempty subset of G. Then, H is a subgroup of G if ab^{-1} is in H whenever a and b are in H. (In additive notation, H is a subgroup if $a - b$ is in H whenever a and b are in H.)

Proof.

1. Associativity: Since the operation of H is the same as that of G, associativity holds.
2. Identity: Since H is nonempty, we may pick some x in H. Let $a = x$ and $b = x$. Then $ab^{-1} = xx^{-1} = e$.
3. Inverses: Let $a = e$ and $b = x$. Then, $ab^{-1} = ex^{-1} = x^{-1}$.
4. Closure: Assume x, y are in H. We have shown that y^{-1} is in H. Let $a = x$ and $b = y^{-1}$. Then $xy = x(y^{-1})^{-1} = ab^{-1}$ is in H.

□

Subgroup Tests

Two-Step Subgroup Test

Theorem: Let G be a group and H a nonempty subset of G. Then, H is a subgroup of G if ab is in H whenever a and b are in H (H is closed under multiplication), and a^{-1} is in H whenever a is in H (inverses).

Proof.

By the one-step subgroup test, it suffices to show that if a and b are in H, then ab^{-1} is in H. Since a^{-1} is in H whenever a is in H, we also know b^{-1} is in H. Thus, ab^{-1} is in H by closure under multiplication.

□

Subgroup Tests

Finite Subgroup Test

Theorem: Let H be a nonempty finite subset of a group G. Then, H is a subgroup of G if H is closed under the operation of G.

Proof.

By the two-step subgroup test, it suffices to show that if a is in H, then a^{-1} is in H. If $a = e$, then $a^{-1} = a$, and we are done.

If $a \neq e$, consider the sequence $a, a^2, \ldots$. By closure, all of these elements belong to H. Since H is finite, not all of these elements will be distinct. Say $a^i = a^j$ and $i > j$. Then, $a^{i-j} = e$; and since $a \neq e$, this means that $i - j > 1$. Thus, $aa^{i-j-1} = a^{i-j} = e$ and therefore, $a^{i-j-1} = a^{-1}$. But, since $i - j - 1 \geq 1$, we know that a^{i-j-1} is in H, and we are done.

□

Primitive Marriage Rules

Example - *Kariera* Society, Second Generation

$$G = \{I, \quad S, \quad D, \quad SD\}$$

Theorem 1: *In the group generated by S and D, every element except I is a complete permutation.*

Theorem 2: *Marriage between relatives of a given kind is always permitted if M = I.*

Theorem 3: *Marriage between relatives of a given kind is never permitted if M is a complete permutation.*

Theorem 4: *$S^{-1}D$ (brother-sister relation) is a complete permutation.*

Permutation Groups

Definitions

Permutation of A*:* A *permutation* of a set *A* is a function from *A* to *A* that is both one-to-one and onto.

Permutation Group of a set A*:* A *permutation group* of a set *A* is a set of permutations of *A* that forms a group under function composition.

Functions

One-to-One and Onto

One-to-one function*:* A function Φ from a set A to itself is called *one-to-one* if for every a_1, a_2 in A, $\Phi(a_1) = \Phi(a_2)$ implies $a_1 = a_2$.

Onto function*:* A function Φ from a set A to itself is called *onto* if for every a_2 in A, there is an a_1 in A such that $\Phi(a_2) = a_1$.

Permutation Groups

Example - *Kariera* Marriage Structure

Permutation Matrices

$$S = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$G = \left\{ S = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},\ D = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},\ D^2 = I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ SD = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right\}$$

Permutation Groups

Example - *Kariera* Marriage Structure

$$G=\left\{S=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix},\quad D=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix},\quad D^2=I=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\quad SD=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\right\}$$

	I	*S*	*D*	*SD*
I	*I*	*S*	*D*	*SD*
S	*S*	*I*	*SD*	*D*
D	*D*	*SD*	*I*	*S*
SD	*SD*	*D*	*S*	*I*

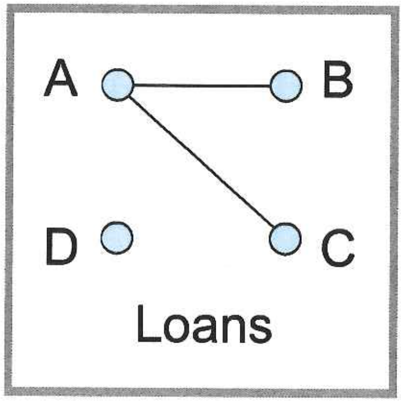
→ *Symmetric - Abelian*

Multiple Network Relations

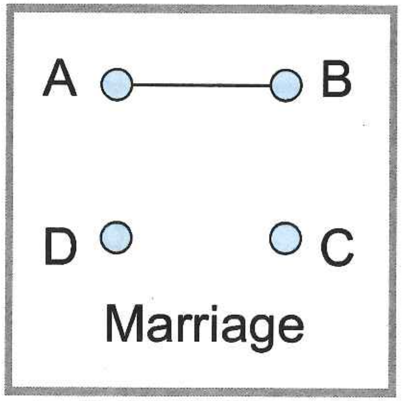
Example

Table 1. Business and Marriage relationships among four individuals

	Business (Loans)				Marriage			
	A	B	C	D	A	B	C	D
A	0	1	1	0	0	1	0	0
B	1	0	0	0	1	0	0	0
C	1	0	0	0	0	0	0	0
D	0	0	0	0	0	0	0	0



$$L = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$



$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

NOTE: *L* and *M* generally NOT **permutation matrices**

Multiple Network Relations

Example

$$L = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad L^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad LM = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$ML = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad M^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad L^2M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad M^2L = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Does this set of matrices constitute a group?
Why or why not?

Multiple Network Relations

Example

Individual *A*'s Row $L' = \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix}$ $M' = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$

Individual *A*'s Row in all Compositions

$$L'L = \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$$

$$M'L = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$$

Table 2. Right multiplication table for individual A

	$L' = \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix}$				$M' = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$			
L	1	0	0	0	1	0	0	0
M	1	0	0	0	1	0	0	0
L^2	0	1	1	0	0	1	1	0
LM	0	1	0	0	0	1	0	0
ML	0	1	1	0	0	1	1	0
M^2	0	1	0	0	0	1	0	0
L^2M	1	0	0	0	1	0	0	0
M^2L	1	0	0	0	1	0	0	0

$A_{L'}$ $A_{M'}$

Local Network Structure

Example - Each Individual as Ego

$$A_{L'} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad B_{L'} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad C_{L'} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad D_{L'} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_{M'} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad B_{M'} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad C_{M'} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad D_{M'} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Local Network Structure

Cumulated Social Roles

- Describes connections with respect to particular actors in the network
- Provides algebraic simplifications of the data
- Preserves connections in the full data set
- Partitions the set of actors into structurally equivalent classes

Breiger, Ronald L. and Philippa E. Pattison. 1986. "Cumulated Social Roles: The Duality of Persons and Their Algebras." *Social Networks*, 8:215-256, 1986.

Southern Women Data

Davis, Gardner, and Gardner (1941)

Davis, Allison, Burleigh B. Gardner, Mary R. Gardner. 1941. *Deep South: A Social Anthropological Study of Caste and Class*. Chicago: The University of Chicago Press.

Goal:

- Examine relation between social class and informal interaction

Data Collection:

- Spent 9 months in Natchez, Mississippi
- Observed 18 women during 14 informal social events ("a day's work behind the counter of a store, a meeting of a women's club, a church supper, a card party, a supper party, a meeting of the PTA, etc")
- Recorded participation using "interviews, the records of participant observers, guest lists, and the newspapers"

Southern Women

Research Questions

- Is the network of Southern women connected through social events?
- Do distinct social groups exist among these Southern women?
- Which of the women are more highly connected than others?

Southern Women Data

Davis, Gardner, and Gardner (1941)

Names of Participants of Group I	Code Numbers and Dates of Social Events Reported in *Old City Herald*													
	(1) 6/27	(2) 3/2	(3) 4/12	(4) 9/26	(5) 2/25	(6) 5/19	(7) 3/15	(8) 9/16	(9) 4/8	(10) 6/10	(11) 2/23	(12) 4/7	(13) 11/21	(14) 8/3
1. Mrs. Evelyn Jefferson	×	×	×	×	×	×		×	×					
2. Miss Laura Mandeville	×	×	×		×	×	×	×						
3. Miss Theresa Anderson		×	×	×	×	×	×	×	×					
4. Miss Brenda Rogers	×		×	×	×	×	×	×						
5. Miss Charlotte McDowd			×	×	×		×							
6. Miss Frances Anderson			×		×	×		×						
7. Miss Eleanor Nye					×	×	×	×						
8. Miss Pearl Oglethorpe						×		×	×					
9. Miss Ruth DeSand					×		×	×	×					
10. Miss Verne Sanderson							×	×	×			×		
11. Miss Myra Liddell								×	×	×		×		
12. Miss Katherine Rogers								×	×	×		×	×	×
13. Mrs. Sylvia Avondale							×	×	×	×		×	×	×
14. Mrs. Nora Fayette						×	×		×	×	×	×	×	×
15. Mrs. Helen Lloyd							×	×		×	×	×		
16. Mrs. Dorothy Murchison								×	×					
17. Mrs. Olivia Carleton									×		×			
18. Mrs. Flora Price									×		×			